

ALMOST LAGRANGIAN OBSTRUCTION

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ABSTRACT. The aim of this paper is to describe the obstruction for an almost Lagrangian fibration to be Lagrangian, a problem which is central to the classification of Lagrangian fibrations and, more generally, to understanding the obstructions to carry out surgery of integrable systems, an idea introduced in [16]. It is shown that this obstruction (namely, the homomorphism \mathcal{D} of Dazord and Delzant [4] and Zung [16]) is related to the cup product in cohomology with local coefficients on the base space B of the fibration. The map is described explicitly and some explicit examples are calculated, thus providing the first examples of non-trivial Lagrangian obstructions.

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1. INTRODUCTION

A fibration $F \hookrightarrow M \rightarrow B$ is said to be *Lagrangian* if M admits a symplectic form ω such that the fibres are Lagrangian. Throughout this paper, the fibres F are taken to be compact and connected; by the Liouville-Mineur-Arnol'd theorem (cf. [1]), this condition implies that the fibres are diffeomorphic to tori. Another consequence of this theorem is that the base space of a Lagrangian fibration is an integral affine manifold, as shown by many authors (*e.g.* [3, 6, 12, 16]). An integral affine structure \mathcal{A} on B is an atlas $\{(U_i, \psi_i)\}$ whose changes of coordinates $\psi_i \circ \psi_j^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ lie in the group

$$\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) := \mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n,$$

where the action of $\mathrm{GL}(n, \mathbb{Z})$ on \mathbb{R}^n is the standard one.

Notation. An integral affine manifold is henceforth denoted by (B, \mathcal{A}) , while $\mathbb{T}^n \hookrightarrow (M, \omega) \rightarrow (B, \mathcal{A})$ denotes a Lagrangian fibration which induces the integral affine structure \mathcal{A} on its base space. The latter is referred to as a Lagrangian fibration over (B, \mathcal{A}) .

Given a Lagrangian fibration over (B, \mathcal{A}) , the atlas \mathcal{A} on the n -dimensional manifold B determines a representation

$$\mathfrak{a} : \pi_1(B) \rightarrow \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n),$$

which arises from identifying $\pi_1(B)$ as the group of deck transformations acting by integral affine diffeomorphisms on the universal cover $(\tilde{B}, \tilde{\mathcal{A}})$, where $\tilde{\mathcal{A}}$ is the integral affine structure on \tilde{B} induced

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by the universal covering $q : \tilde{B} \rightarrow B$ (cf. [2]). The homomorphism \mathfrak{a} is known as the affine holonomy of the integral affine manifold (B, \mathcal{A}) , while the composite

$$\mathfrak{l} := \text{Lin} \circ \mathfrak{a} : \pi_1(B) \rightarrow \text{Aff}_{\mathbb{Z}}(\mathbb{R}^n) \rightarrow \text{GL}(n, \mathbb{Z})$$

is called the linear holonomy of (B, \mathcal{A}) . On the other hand, associated to a Lagrangian fibration over (B, \mathcal{A}) is a homomorphism

$$\rho : \pi_1(B) \rightarrow \text{GL}(n, \mathbb{Z}),$$

called the *monodromy* of the fibration (cf. [6]). The monodromy and linear holonomy are related by

$$(1) \quad \mathfrak{l} = \rho^{-T},$$

where $-T$ denotes inverse transposed. Conversely, any n -dimensional integral affine manifold (B, \mathcal{A}) with linear holonomy \mathfrak{l} is the base space of a Lagrangian fibration $\mathbb{T}^n \hookrightarrow (M, \omega) \rightarrow (B, \mathcal{A})$ with monodromy representation \mathfrak{l}^{-T} .

Another consequence of the Liouville-Mineur-Arnol'd theorem also is that the structure group of a Lagrangian fibration over an n -dimensional manifold reduces to

$$\text{Aff}_{\mathbb{Z}}(\mathbb{R}^n / \mathbb{Z}^n) := \text{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n / \mathbb{Z}^n.$$

This observation implies there are only two topological invariants associated to a Lagrangian fibration, namely the aforementioned monodromy ρ , and a cohomology class $[c] \in H^2(B; \mathbb{Z}_{\rho}^n)$ called the Chern class, which is the obstruction to the existence of a section (cf. [6, 12]). Note that $[c]$ lies in the cohomology theory with local coefficients in the module $\pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$ twisted by the representation ρ (cf. [4, 6, 12, 16]). Given an integral affine manifold (B, \mathcal{A}) with linear holonomy \mathfrak{l} , the set of (isomorphism classes of) Lagrangian fibrations over (B, \mathcal{A}) (and, thus, with monodromy \mathfrak{l}^{-T}) is in 1-1 correspondence with a subgroup

$$R \leq H^2(B; \mathbb{Z}_{\mathfrak{l}^{-T}}^n).$$

It is natural to ask what subgroup R is: work of Dazord and Delzant in [4] and of Zung in [16] proves that there exists a homomorphism

$$\mathcal{D} : H^2(B; \mathbb{Z}_{\mathfrak{l}^{-T}}^n) \rightarrow H^3(B; \mathbb{R})$$

with $R = \ker \mathcal{D}$. This map represents the obstruction to construct an appropriate symplectic form on the total space of an *almost Lagrangian fibration* over (B, \mathcal{A}) (cf. Definition 1). However, there is no explicit general description of this homomorphism in the literature.

The aim of this paper is to relate the homomorphism \mathcal{D} to the cohomology of B in the case of regular Lagrangian fibrations and to carry out some explicit calculations. Dazord and Delzant in [4] show that, when the monodromy representation is trivial, the homomorphism \mathcal{D} is given by the *cup product* on the singular cohomology ring of B . The main result of this paper generalises this description.

Main Result. The homomorphism \mathcal{D} arises from taking twisted cup product on B (*i.e.* cup product in cohomology with local coefficients).

For a fixed integral affine manifold (B, \mathcal{A}) , the homomorphism \mathcal{D} distinguishes those almost Lagrangian fibrations over (B, \mathcal{A}) which are actually Lagrangian and those which are not; the latter are called *fake*. The local structure of almost and fake Lagrangian fibrations has been studied by Fassò and Sansonetto [7] in the context of a generalised notion of Liouville integrability. Elements of R are called *realisable*, the terminology coming from the theory of symplectic realisations of

Poisson manifolds (cf. [4, 14]). The above Main Result is important in the classification of Lagrangian fibration over manifolds of dimension greater than or equal to 3, in so far as it provides an algorithm to compute the subgroup of realisable classes. Furthermore, the constructions in this paper are related to the idea of *integrable surgery*, introduced in [16], where two or more completely integrable Hamiltonian systems are glued together to yield a new one.

The structure of the paper is as follows. The notion of *almost Lagrangian fibrations* over an integral affine manifold (B, \mathcal{A}) is introduced in Section 2, following ideas in [7]. These fibrations are the right ‘candidates’ to be Lagrangian, as they share with their Lagrangian counterparts all topological invariants and constructs. Section 3 deals with the proof of the Main Result and is divided into two parts. The first sets up some notation and describes the obstruction in the case of trivial linear holonomy of (B, \mathcal{A}) (cf. [4]). The general case is presented in Section 3.2: the homomorphism \mathcal{D} is described in detail and shown to coincide with the twisted cup product on B . Moreover, an explicit algorithm to compute R in general is provided. This is a non-trivial task since the homomorphisms involved in the definition of \mathcal{D} are defined on simplicial spaces, which are not easily dealt with from a computational point of view. In Section 4, there are some examples related to the classification of Lagrangian fibrations. In particular, the algorithm outlined in Section 3 is applied to some concrete examples of integral affine manifolds.

2. ALMOST LAGRANGIAN FIBRATIONS

Throughout this section, fix an integral affine manifold (B, \mathcal{A}) with linear holonomy $\mathfrak{l} : \pi_1(B) \rightarrow \mathrm{GL}(n, \mathbb{Z})$. Denote by $\pi : M \rightarrow B$ the projection map of a Lagrangian fibration $\mathbb{T}^n \hookrightarrow (M, \omega) \rightarrow (B, \mathcal{A})$. The Liouville-Arnol’d-Mineur theorem (cf. [1]) implies that there exists a good (in the sense of Leray) open cover $\mathcal{U} = \{U_i\}$ of B by trivialising neighbourhoods such that $\pi^{-1}(U_i)$ has coordinates $(\mathbf{x}_i, \mathbf{t}_i)$ (called local *action-angle* coordinates) with the following properties

- the action coordinates \mathbf{x}_i define the integral affine structure \mathcal{A} , while the angle coordinates \mathbf{t}_i define integral affine coordinates on the fibres;
- for each i , the restriction $\omega_i = \omega|_{\pi^{-1}(U_i)}$ takes the form

$$\omega_i = \sum_{l=1}^n dx_i^l \wedge dt_i^l;$$

- the transition functions ϕ_{ji} are given by

$$(2) \quad \begin{aligned} \phi_{ji} : (U_i \cap U_j) \times \mathbb{T}^n &\rightarrow (U_i \cap U_j) \times \mathbb{T}^n \\ (\mathbf{x}_i, \mathbf{t}_i) &\mapsto (A_{ji}\mathbf{x}_i + \mathbf{c}_{ji}, A_{ji}^{-T}\mathbf{t}_i + \mathbf{g}_{ji}(\mathbf{x}_i)) \end{aligned}$$

where $A_{ji} \in \mathrm{GL}(n, \mathbb{Z})$ and $\mathbf{c}_{ji} \in \mathbb{R}^n$ are constant, and $\mathbf{g}_{ji} : U_{ji} \rightarrow \mathbb{T}^n$ is a local function constrained by the fact that

$$\phi_{ji}^* \omega_j = \omega_i$$

for all i, j .

The above necessary conditions for a Lagrangian fibration over (B, \mathcal{A}) and the results of [7] motivate the following definition.

Definition 1. A fibration $\mathbb{T}^n \hookrightarrow M \rightarrow (B, \mathcal{A})$ is said to be *almost Lagrangian* if there exists a good open cover $\mathcal{U} = \{U_i\}$ of B such that

- there exist trivialisations $\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{T}^n$ inducing local coordinates $(\mathbf{x}_i, \mathbf{t}_i)$;
- the local coordinates \mathbf{x}_i are integral affine coordinates on (B, \mathcal{A}) ;

- the transition functions $\phi_{ji} = \varphi_j \circ \varphi_i^{-1}$ are of the form

$$(3) \quad \phi_{ji}(\mathbf{x}_i, \mathbf{t}_i) = (A_{ji}\mathbf{x}_i + \mathbf{c}_{ji}, A_{ji}^{-T}\mathbf{t}_i + \mathbf{g}_{ji}(\mathbf{x}_i))$$

where $A_{ji} \in \mathrm{GL}(n, \mathbb{Z})$ and $\mathbf{c}_{ji} \in \mathbb{R}^n$ are constant, $\mathbf{g}_{ji} : U_{ji} \rightarrow \mathbb{T}^n$ is a local function.

Remark 1.

- An almost Lagrangian fibration $\mathbb{T}^n \hookrightarrow M \rightarrow (B, \mathcal{A})$ has structure group $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$. Thus the topological classification of almost Lagrangian fibrations coincides with that of Lagrangian fibrations (cf. [6]) and it makes sense to consider the monodromy and Chern class of such fibrations;
- If ρ denotes the monodromy representation of an almost Lagrangian fibration $\mathbb{T}^n \hookrightarrow M \rightarrow (B, \mathcal{A})$, then

$$\rho = \mathfrak{l}^{-T},$$

where \mathfrak{l} denotes the linear holonomy of (B, \mathcal{A}) ;

- For a fixed almost Lagrangian fibration, each $\pi^{-1}(U_i)$ admits a local symplectic form

$$(4) \quad \omega_i = \sum_{l=1}^n dx_i^l \wedge dt_i^l,$$

where $(\mathbf{x}_i, \mathbf{t}_i)$ are local coordinates as in Definition 1. An almost Lagrangian fibration is Lagrangian if and only if it is possible to choose the transition functions ϕ_{ji} so that

$$\phi_{ji}^* \omega_j = \omega_i.$$

- In analogy with the Lagrangian case, $\mathbb{T}^n \hookrightarrow M \rightarrow (B, \mathcal{A})$ denotes an almost Lagrangian fibration constructed from the integral affine structure \mathcal{A} on B . Such fibrations are in 1-1 correspondence with the set of \mathbb{T}^n -bundles with structure group $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n/\mathbb{Z}^n)$ whose monodromy equals \mathfrak{l}^{-T} .

The period lattice bundle associated to an integral affine manifold (B, \mathcal{A}) plays an important role in the study and construction of almost Lagrangian fibrations.

Definition 2. Let \mathbf{x}_i be local integral affine coordinates defined on subsets U_i of (B, \mathcal{A}) . The *period lattice bundle* $P_{(B, \mathcal{A})} \subset T^*B$ associated to (B, \mathcal{A}) is the discrete subbundle (with fibre \mathbb{Z}^n) which is locally defined by

$$P_{(B, \mathcal{A})}|_{U_i} := \{(\mathbf{x}_i, \mathbf{v}_i) \in T^*U_i : \mathbf{v}_i \in \mathbb{Z}\langle dx_i^1, \dots, dx_i^n \rangle\}.$$

Remark 2. Let $\mathbb{T}^n \hookrightarrow M \rightarrow (B, \mathcal{A})$ be an almost Lagrangian fibration. Just as in the Lagrangian case, the bundle obtained by replacing the torus fibres with their first homology groups $H_1(\mathbb{T}^n; \mathbb{Z})$ is isomorphic to the period lattice bundle associated to (B, \mathcal{A}) . The isomorphism is defined locally using the symplectic forms ω_i of Remark 1, and it sends the differentials dx_i^l to the homology cycle generated by the time-1 flow of the Hamiltonian vector field of the function x_i^l (cf. [6]).

The inclusion $P_{(B, \mathcal{A})} \hookrightarrow T^*B$ defines an injective morphism of sheaves

$$(5) \quad \mathcal{P}_{(B, \mathcal{A})} \hookrightarrow \mathcal{Z}^1(B)$$

where $\mathcal{P}_{(B, \mathcal{A})}, \mathcal{Z}^1(B)$ denote the sheaves of sections of the period lattice bundle $P_{(B, \mathcal{A})} \rightarrow B$ and of closed sections of the cotangent bundle $T^*B \rightarrow B$ respectively (cf. [4, 16]). Recall that the Chern class of an almost Lagrangian fibration over (B, \mathcal{A}) lies in the group

$$H^2(B; \mathcal{P}_{(B, \mathcal{A})}) \cong H^2(B; \mathbb{Z}_{\mathfrak{l}^{-T}}^n),$$

as shown in [4, 6]. It is important to note that the above isomorphism is induced by the isomorphism of \mathbb{Z}^n -bundles described in Remark 2. Henceforth, the local coefficient system with values

in \mathbb{Z}^n and twisted by the representation \mathbb{I}^{-T} is identified with the stalk of the sheaf of sections of the period lattice bundle $P_{(B, \mathcal{A})}$.

The morphism of equation (5) induces a homomorphism of cohomology groups

$$(6) \quad H^2(B; \mathcal{P}_{(B, \mathcal{A})}) \rightarrow H^2(B; \mathcal{Z}^1(B)).$$

If $\mathcal{C}^\infty(B)$ denotes the sheaf of smooth functions on B , there is a short exact sequence of sheaves

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{C}^\infty(B) \xrightarrow{d} \mathcal{Z}^1(B) \longrightarrow 0,$$

where d denotes the standard exterior differential; since $\mathcal{C}^\infty(B)$ is a fine sheaf, the induced long exact sequence in cohomology yields an isomorphism

$$(7) \quad H^2(B; \mathcal{Z}^1(B)) \cong H^3(B; \mathbb{R}).$$

Definition 3 (Dazord and Delzant [4]). The composition of the homomorphism of equation (6) and the isomorphism of equation (7) yields a homomorphism

$$\mathcal{D} : H^2(B; \mathcal{P}_{(B, \mathcal{A})}) \rightarrow H^3(B; \mathbb{R}),$$

called the *Dazord-Delzant* homomorphism.

Remark 3 (Dazord and Delzant [4]). For an almost Lagrangian fibration over (B, \mathcal{A}) with transition functions ϕ_{ji} as in Definition 1, a Čech cocycle representing $\mathcal{D}[c]$ is given by

$$\kappa_{ji} = \phi_{ji}^* \omega_j - \omega_i.$$

Remark 3 lies at the heart of the proof of the following theorem, stated below without proof as it is a known result.

Theorem 1 (Dazord and Delzant [4]). *The subgroup of realisable Chern classes for a given integral affine manifold (B, \mathcal{A}) is given by $\ker \mathcal{D}$.*

3. THE DAZORD-DELZANT HOMOMORPHISM AND EQUIVARIANT CUP PRODUCTS

3.1. The case with trivial monodromy. Let (B, \mathcal{A}) be an integral affine manifold with trivial linear holonomy. The period lattice bundle $P_{(B, \mathcal{A})}$ admits a global frame of closed forms $\{\theta^1, \dots, \theta^n\}$ which, locally, are given by the differentials of integral affine coordinates (cf. [2]). The set of isomorphism classes of almost Lagrangian bundles over (B, \mathcal{A}) is in 1-1 correspondence with elements of the cohomology group $H^2(B; \mathbb{Z}^n)$. The universal coefficient theorem and the fact that \mathbb{Z}^n is a free module imply that there is an isomorphism

$$(8) \quad H^2(B; \mathbb{Z}^n) \cong H^2(B; \mathbb{Z}) \otimes \mathbb{Z}^n$$

(cf. [13]). The above coefficient system \mathbb{Z}^n is given by the stalk of the sheaf of sections $\mathcal{P}_{(B, \mathcal{A})}$ of the period lattice bundle $P_{(B, \mathcal{A})}$, which, in this case, corresponds to integral linear combinations of $\theta^1, \dots, \theta^n$. Thus elements of $H^2(B; \mathbb{Z}^n)$ are of the form

$$\sum_{p, q} n_{pq} [\beta_p] \otimes \theta^q,$$

where $n_{pq} \in \mathbb{Z}$ and $\{[\beta_p]\}$ form a basis of $H^2(B; \mathbb{Z})$. The following theorem, stated below without proof, describes the Dazord-Delzant homomorphism in the case of trivial linear holonomy of the base space (B, \mathcal{A}) .

Theorem 2 (Dazord-Delzant [4]). *The map $\mathcal{D} : H^2(B; \mathbb{Z}^n) \rightarrow H^3(B; \mathbb{R})$ is given by*

$$(9) \quad \sum_{p,q} n_{pq} [\beta_p] \otimes \theta^q \mapsto \left[\sum_{p,q} n_{pq} \beta_p^{\mathbb{R}} \cup \theta^q \right]$$

where $\beta_p^{\mathbb{R}}$ is a 2-cocycle with values in the real numbers representing the image of a cocycle β_p representing $[\beta_p]$ under the map $H^2(B; \mathbb{Z}) \rightarrow H^2(B; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong H^2(B; \mathbb{R})$.

3.2. The general case. Theorem 2 shows that the map \mathcal{D} is related to cup product in the real cohomology of an integral affine manifold (B, \mathcal{A}) with trivial linear holonomy. In general, the situation is not as simple. Let (B, \mathcal{A}) be an integral affine manifold with linear holonomy \mathfrak{l} ; $H^2(B; \mathbb{Z}_{[-T]}^n)$ denotes the set of isomorphism classes of almost Lagrangian fibrations over (B, \mathcal{A}) . The universal coefficient theorem for cohomology with local coefficients in [10] gives the following short exact sequence

$$0 \rightarrow H^2(B; \mathbb{Z}[\pi]_{\text{taut}}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}^n \rightarrow H^2(B; \mathbb{Z}_{\rho}^n) \rightarrow \text{Tor}(H^3(B; \mathbb{Z}[\pi]_{\text{taut}}), \mathbb{Z}^n) \rightarrow 0$$

where $\pi = \pi_1(B)$, $\mathbb{Z}[\pi]$ denotes the group ring of π and $\text{taut} : \pi \rightarrow \text{Aut}(\mathbb{Z}[\pi])$ defines the tautological representation of π on $\mathbb{Z}[\pi]$ given by left multiplication. In general, neither $H^3(B; \mathbb{Z}[\pi]_{\text{taut}})$ nor \mathbb{Z}^n are free $\mathbb{Z}[\pi]$ -modules and so there is no equivariant equivalent of the isomorphism in equation (8). In what follows, a candidate for the map \mathcal{D} is suggested and then it is proved to be the correct one in Theorem 3.

Throughout this section, fix an integral affine manifold (B, \mathcal{A}) with linear holonomy \mathfrak{l} and let π denote its fundamental group. Let $\mathbb{T}^n \hookrightarrow M \rightarrow (B, \mathcal{A})$ be an almost Lagrangian fibration with Chern class $[c] \in H^2(B; \mathbb{Z}_{[-T]}^n)$. Let $q : \tilde{B} \rightarrow B$ be the universal covering of B , and let $\tilde{\mathcal{A}}$ denote the induced integral affine structure on \tilde{B} via q (cf. [2]).

Remark 4. The Chern class $[c]$ of the fibration $M \rightarrow B$ is pulled back to the Chern class $[\tilde{c}] = q^*[c] \in H^2(\tilde{B}; \mathbb{Z}^n)$ of $q^*M \rightarrow (\tilde{B}, \tilde{\mathcal{A}})$ by functoriality of the Chern class (cf. [12]). A cocycle representing $[c]$ is a cochain in $\text{hom}_{\mathbb{Z}[\pi]}(C_2(\tilde{B}); \mathbb{Z}^n)$, where \mathbb{Z}^n is a $\mathbb{Z}[\pi]$ -module via the monodromy representation \mathfrak{l}^{-T} . In particular, if c is a such a cocycle, it is just an *equivariant* cocycle representing $[\tilde{c}]$, since $\text{hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{B}); \mathbb{Z}^n)$ is a subcomplex of complex $\text{hom}_{\mathbb{Z}}(C_*(\tilde{B}); \mathbb{Z}^n)$ (cf. [15]).

Let $P_{(B, \mathcal{A})}, P_{(\tilde{B}, \tilde{\mathcal{A}})}$ denote the period lattice bundles associated to (B, \mathcal{A}) and $(\tilde{B}, \tilde{\mathcal{A}})$ respectively; there is a commutative diagram

$$(10) \quad \begin{array}{ccc} \tilde{P}_{(\tilde{B}, \tilde{\mathcal{A}})} & \hookrightarrow & T^*\tilde{B} \\ \downarrow Q & & \downarrow Q \\ P_{(B, \mathcal{A})} & \hookrightarrow & T^*B, \end{array}$$

where $Q : T^*\tilde{B} \cong q^*T^*B \rightarrow T^*B$ is induced by the quotient map q . The above diagram implies that there exists a π -equivariant global frame of closed forms $\{\tilde{\theta}^1, \dots, \tilde{\theta}^n\}$ of $T^*\tilde{B}$; thus the fibre \mathbb{Z}^n of the period lattice bundle $P_{(B, \mathcal{A})} \rightarrow B$ can be identified with the fibre of $\tilde{P}_{(\tilde{B}, \tilde{\mathcal{A}})} \rightarrow \tilde{B}$. Therefore the inclusion $\tilde{P}_{(\tilde{B}, \tilde{\mathcal{A}})} \subset T^*\tilde{B}$ induces a homomorphism

$$\chi : \text{hom}_{\mathbb{Z}}(C_2(\tilde{B}); \mathbb{Z}^n) \rightarrow \text{hom}_{\mathbb{Z}}(C_2(\tilde{B}); \text{hom}_{\mathbb{Z}}(C_1(\tilde{B}); \mathbb{R})),$$

which maps the subgroup $\text{hom}_{\mathbb{Z}[\pi]}(C_2(\tilde{B}); \mathbb{Z}^n)$ to π -equivariant homomorphisms.

Corollary 1. *The following diagram commutes*

$$(11) \quad \begin{array}{ccc} \mathrm{hom}_{\mathbb{Z}[\pi]}(C_2(\tilde{B}); \mathbb{Z}^n) & \xrightarrow{\iota} & \mathrm{hom}_{\mathbb{Z}}(C_2(\tilde{B}); \mathbb{Z}^n) \\ \downarrow \chi & & \downarrow \chi \\ \mathrm{hom}_{\mathbb{Z}[\pi]}(C_2(\tilde{B}); \mathrm{hom}_{\mathbb{Z}}(C_1(\tilde{B}); \mathbb{R})) & \xrightarrow{\iota} & \mathrm{hom}_{\mathbb{Z}}(C_2(\tilde{B}); \mathrm{hom}_{\mathbb{Z}}(C_1(\tilde{B}); \mathbb{R})) \end{array}$$

Proof. The result follows from the commutativity of the diagram in equation (10). \square

Remark 5. In order for $\mathrm{hom}_{\mathbb{Z}[\pi]}(C_2(\tilde{B}); \mathrm{hom}_{\mathbb{Z}}(C_1(\tilde{B}); \mathbb{R}))$ to be a $\mathbb{Z}[\pi]$ -module, fix the action of π on $C_2(\tilde{B})$ to be on the left, while the action on $C_1(\tilde{B})$ to be on the right, so that $\mathrm{hom}_{\mathbb{Z}}(C_1(\tilde{B}); \mathbb{R})$ is a left $\mathbb{Z}[\pi]$ -module. While it is a technical point, it is important for the proof of Lemma 1.

Let c be a cocycle representing the Chern class $[c]$ of the almost Lagrangian fibration. The differential D on the complex

$$\mathrm{hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{B}); \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}); \mathbb{R}))$$

is given by the graded sum of the differentials ∂ and δ on the complexes $C_*(\tilde{B})$ and $\mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}); \mathbb{R})$ respectively. Then

$$D\chi(c) = 0,$$

since $\chi(c)$ takes values in *closed* 1-forms. The chain isomorphism (cf. [5])

$$\Psi : \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}); \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}); \mathbb{R})) \rightarrow \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}) \otimes_{\mathbb{Z}} C_*(\tilde{B}); \mathbb{R})$$

defined by

$$\Psi(g)(x \otimes y) = (f(y))(x)$$

for all $g \in \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}); \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}); \mathbb{R}))$, $x \in C_*(\tilde{B})$ and $y \in C_*(\tilde{B})$, preserves the $\mathbb{Z}[\pi]$ -module structures defined in Remark 5 and, thus, defines a chain isomorphism

$$\Psi : \mathrm{hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{B}); \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}); \mathbb{R})) \rightarrow \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}) \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{B}); \mathbb{R}).$$

Therefore there is a commutative diagram

$$(12) \quad \begin{array}{ccc} \mathrm{hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{B}); \mathrm{hom}_{\mathbb{Z}}(C_1(\tilde{B}); \mathbb{R})) & \xrightarrow{\iota} & \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}); \mathrm{hom}_{\mathbb{Z}}(C_1(\tilde{B}); \mathbb{R})) \\ \downarrow \Psi & & \downarrow \Psi \\ \mathrm{hom}_{\mathbb{Z}}(C_1(\tilde{B}) \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{B}); \mathbb{R}) & \xrightarrow{\iota} & \mathrm{hom}_{\mathbb{Z}}(C_1(\tilde{B}) \otimes_{\mathbb{Z}} C_*(\tilde{B}); \mathbb{R}). \end{array}$$

In light of the above commutative diagram, $\Psi \circ \chi(c)$ can be seen as an equivariant element of $\mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}) \otimes_{\mathbb{Z}} C_*(\tilde{B}); \mathbb{R})$. Consider the composition

$$\Phi : \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}) \otimes_{\mathbb{Z}} C_*(\tilde{B}); \mathbb{R}) \rightarrow \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B} \times \tilde{B}); \mathbb{R}) \rightarrow \mathrm{hom}_{\mathbb{Z}}(C_*(\tilde{B}); \mathbb{R}),$$

where the first homomorphism is induced by the Alexander-Whitney map

$$A : C_*(\tilde{B} \times \tilde{B}) \rightarrow C_*(\tilde{B}) \otimes_{\mathbb{Z}} C_*(\tilde{B})$$

and the second by the diagonal map

$$\tilde{\Delta} : \tilde{B} \rightarrow \tilde{B} \times \tilde{B}.$$

It is standard that these maps can be made into equivariant simplicial chain maps (cf. [13]), *i.e.* they can be chosen to preserve the underlying $\mathbb{Z}[\pi]$ -module structures. Therefore there is a well-defined map

$$(13) \quad \Phi \circ \Psi \circ \chi : \mathrm{hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{B}); \mathbb{Z}^n) \rightarrow \mathrm{hom}_{\mathbb{Z}}(C_{*+1}(\tilde{B}); \mathbb{R}),$$

which descends to cohomology. This is the candidate for \mathcal{D} . Note that the map $\Phi \circ \Psi \circ \chi$ on standard cochains on \tilde{B} is simply given by the cup product by definition of the various homomorphisms involved (cf. [15]).

Definition 4. The map of equation (13) is called the *twisted cup product* on (B, \mathcal{A}) .

The following lemma proves that the image of the twisted cup product lies in the π -invariant forms on \tilde{B} , *i.e.* it defines a form on B .

Lemma 1. *Let c be a cocycle representing the Chern class $[c]$ of the fixed almost Lagrangian fibration $\mathbb{T}^n \hookrightarrow M \rightarrow (B, \mathcal{A})$. For all $\gamma \in \pi$,*

$$\gamma \cdot (\Phi \circ \Psi \circ \chi(c)) = \Phi \circ \Psi \circ \chi(c)$$

Proof. To simplify notation, set $\chi(c) = f$. Fix $\gamma \in \pi$ and let z be a singular 3-simplex taking values in \tilde{B} . Then

$$(14) \quad \begin{aligned} (\Phi \circ \Psi)(f)(z) &= (\Psi(f))((z \circ \lambda^1) \otimes_{\mathbb{Z}\pi} (z \circ \mu^2)) \\ &= f(z \circ \mu^2)(z \circ \lambda^1), \end{aligned}$$

where the first equality follows from noticing that $\tilde{\Phi}$ is obtained by composing the pullbacks of the diagonal map and the Alexander-Whitney map on \tilde{B} . The map $\lambda^1 : \Delta^1 \rightarrow \Delta^3$ is defined on vertices of the 1-simplex by $\lambda^1 e_i = e_i$ for $i = 0, 1$, while $\mu^2 : \Delta^2 \rightarrow \Delta^3$ is defined on vertices of the 2-simplex by $\mu^2 e_j = e_{1+j}$ for $j = 0, 1, 2$ (cf. [13]). On the other hand,

$$\begin{aligned} (\gamma \cdot (\Phi \circ \Psi)(f))(z) &= (\Phi \circ \Psi)(f)(\gamma \cdot z) \\ &= (\Psi(f))(((\gamma \cdot z) \circ \lambda^1) \otimes_{\mathbb{Z}\pi} ((\gamma \cdot z) \circ \mu^2)). \end{aligned}$$

By definition

$$(15) \quad (\gamma \cdot z) \circ \lambda^1(t) = \gamma((z \circ \lambda^1)(t)) = ((z \circ \lambda^1)(t)) \cdot \gamma^{-1}$$

where the last equality follows from the fact that $C_1(\tilde{B})$ is a *right* $\mathbb{Z}[\pi]$ -module (cf. Remark 5). Similarly,

$$(16) \quad (\gamma \cdot z) \circ \mu^2(t) = \gamma((z \circ \mu^2)(t)) = \gamma \cdot ((z \circ \mu^2)(t)).$$

Combining equations (15) and (16)

$$(17) \quad \begin{aligned} (\Psi(f))(((\gamma \cdot z) \circ \lambda^1) \otimes_{\mathbb{Z}\pi} ((\gamma \cdot z) \circ \mu^2)) &= (\Psi(f))((z \circ \lambda^1) \cdot \gamma^{-1} \otimes_{\mathbb{Z}\pi} (\gamma \cdot (z \circ \mu^2))) \\ &= (\Psi(f))((z \circ \lambda^1) \otimes_{\mathbb{Z}\pi} (z \circ \mu^2)) \\ &= f(z \circ \mu^2)(z \circ \lambda^1), \end{aligned}$$

where the penultimate equality follows by properties of the tensor product. Comparing equations (14) and (17) yields the required result. \square

Lemma 1 shows that $\Phi \circ \Psi \circ \chi$ defines a homomorphism of cohomology groups

$$H^2(B; \mathbb{Z}_{l^{-T}}^n) \rightarrow H^3(B; \mathbb{R}).$$

It remains to show that it coincides with the map \mathcal{D} . Let c be a cocycle representing the Chern class $[c]$ of the almost Lagrangian fibration $\mathbb{T}^n \hookrightarrow M \rightarrow (B, \mathcal{A})$; by Remark 4, considering c as a standard cocycle on \tilde{B} yields a cocycle representing $q^*[c]$, *i.e.* the Chern class of the almost Lagrangian bundle obtained by pulling back along the universal covering q . The commutative diagrams of Corollary 1 and equation (12) imply that

$$(18) \quad \Psi \circ \Phi \circ \chi(c) = \Psi \circ \Phi \circ \chi(\iota(c)),$$

where

$$\iota : \text{hom}_{\mathbb{Z}[\pi]}(C_2(\tilde{B}); \mathbb{Z}^n) \hookrightarrow \text{hom}_{\mathbb{Z}}(C_2(\tilde{B}); \mathbb{Z}^n)$$

denotes inclusion. The homomorphism

$$\Psi \circ \Phi : \text{hom}_{\mathbb{Z}}(C_2(\tilde{B}); \text{hom}_{\mathbb{Z}}(C_1(\tilde{B}); \mathbb{R})) \rightarrow \text{hom}_{\mathbb{Z}}(C_3(\tilde{B}); \mathbb{R})$$

equals cup product by definition. Therefore, on the level of cohomology groups, it equals the Dazord-Delzant homomorphism $\tilde{\mathcal{D}}$ for the integral affine manifold $(\tilde{B}, \tilde{\mathcal{A}})$ by Theorem 2. Since ι induces the pullback map q^* on cohomology, it follows that

$$(19) \quad \Phi \circ \Psi \circ \chi = \tilde{\mathcal{D}} \circ q^* : H^2(B; \mathbb{Z}_{|-T}^n) \rightarrow H^3(\tilde{B}; \mathbb{R}).$$

Lemma 2. *The following diagram commutes*

$$\begin{array}{ccc} H^2(B; \mathbb{Z}_{|-T}^n) & \xrightarrow{\mathcal{D}} & H^3(B; \mathbb{R}) \\ \downarrow q^* & & \downarrow q^* \\ H^2(\tilde{B}; \mathbb{Z}^n) & \xrightarrow{\tilde{\mathcal{D}}} & H^3(\tilde{B}; \mathbb{R}) \end{array}$$

Proof. There is a commutative diagram of sheaves

$$(20) \quad \begin{array}{ccc} \mathcal{P}_{(B, \mathcal{A})} & \xrightarrow{j} & \mathcal{Z}^1(B) \\ \downarrow q^* & & \downarrow q^* \\ \tilde{\mathcal{P}}_{(\tilde{B}, \tilde{\mathcal{A}})} & \xrightarrow{j} & \mathcal{Z}^1(\tilde{B}) \end{array}$$

where $\mathcal{P}_{(B, \mathcal{A})}, \tilde{\mathcal{P}}_{(\tilde{B}, \tilde{\mathcal{A}})}$ ($\mathcal{Z}^1(B), \mathcal{Z}^1(\tilde{B})$) are the sheaves of sections of the period lattice bundles $P_{(B, \mathcal{A})} \rightarrow B, \tilde{P}_{(\tilde{B}, \tilde{\mathcal{A}})} \rightarrow \tilde{B}$ (sheaves of closed sections of the cotangent bundles $T^*B \rightarrow B, T^*\tilde{B} \rightarrow \tilde{B}$) respectively. The commutativity of the above diagram follows from equation (10). Equation (20) induces a commutative diagram of cohomology groups

$$\begin{array}{ccc} H^2(B; \mathcal{P}_{(B, \mathcal{A})}) & \xrightarrow{\mathfrak{D}} & H^2(B; \mathcal{Z}^1(B)) \\ \downarrow q^* & & \downarrow q^* \\ H^2(\tilde{B}; \tilde{\mathcal{P}}_{(\tilde{B}, \tilde{\mathcal{A}})}) & \xrightarrow{\tilde{\mathfrak{D}}} & H^2(\tilde{B}; \mathcal{Z}^1(\tilde{B})). \end{array}$$

There is a commutative ladder of short exact sequences of sheaves

$$(21) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{C}^\infty(B) & \xrightarrow{d} & \mathcal{Z}^1(B) \longrightarrow 0 \\ & & \downarrow q^* & & \downarrow q^* & & \downarrow q^* \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{C}^\infty(\tilde{B}) & \xrightarrow{d} & \mathcal{Z}^1(\tilde{B}) \longrightarrow 0; \end{array}$$

since both sheaves of smooth functions $\mathcal{C}^\infty(B)$ and $\mathcal{C}^\infty(\tilde{B})$ are fine, it follows that there are isomorphisms

$$\begin{aligned} H^2(B; \mathcal{Z}^1(B)) &\cong H^3(B; \mathbb{R}) \\ H^2(\tilde{B}; \mathcal{Z}^1(\tilde{B})) &\cong H^3(\tilde{B}; \mathbb{R}) \end{aligned}$$

which commute with the natural maps q^* induced by the universal covering $q : \tilde{B} \rightarrow B$. Applying these isomorphisms to the commutative diagram of equation (21), the result follows. \square

Lemma 2 states that $\tilde{\mathcal{D}} \circ q^* = q^* \circ \mathcal{D}$. It follows from equation (19) that

$$(22) \quad \Phi \circ \Psi \circ \chi = \tilde{\mathcal{D}} \circ q^* = q^* \circ \mathcal{D}$$

Lemma 1 shows that the image of the map $\Phi \circ \Psi \circ \chi$ lies precisely in the subgroup of π -invariant closed 3-forms on \tilde{B} . Such forms are in 1 – 1 correspondence with cohomology classes of 3-forms on B . Denote by $[\Phi \circ \Psi \circ \chi(c)]_B$ the cohomology class of the 3-form on B obtained from $\Phi \circ \Psi \circ \chi(c)$. Therefore the following theorem, which contains the main result of the paper, holds.

Theorem 3. *If c denotes a cocycle representing $[c] \in H^2(B; \mathbb{Z}_\rho^n)$*

$$[\Phi \circ \Psi \circ \chi(c)]_B = \mathcal{D}[c]$$

for all $[c] \in H^2(B; \mathbb{Z}_\rho^n)$. In other words, the Dazord-Delzant homomorphism \mathcal{D} is given by taking the twisted cup product.

Remark 6. It is not necessary to use the universal cover \tilde{B} in the above discussion. It is enough to consider an integral affine covering $(\hat{B}, \hat{\mathcal{A}})$ of (B, \mathcal{A}) corresponding to the normal subgroup $\ker \iota^{-T}$. By construction, $\hat{\mathcal{A}}$ has trivial linear holonomy and so the above considerations can be applied. More generally, Lemma 2 can also be applied to any integral affine regular covering $(\tilde{B}, \tilde{\mathcal{A}})$ of (B, \mathcal{A}) to show an analogous relation between the maps $\mathcal{D}, \tilde{\mathcal{D}}$.

The above discussion provides an explicit algorithm to compute \mathcal{D} for any integral affine manifold (B, \mathcal{A}) with linear holonomy \mathbf{l} . Given an almost Lagrangian fibration $\mathbb{T}^n \hookrightarrow M \rightarrow (B, \mathcal{A})$ with Chern class $[c]$, let c be an equivariant cocycle representing $[c]$. Fix a cell decomposition of B (which is assumed here to be a CW complex without loss of generality) and a \mathbf{l}^{-T} -equivariant global frame $\{\tilde{\theta}^1, \dots, \tilde{\theta}^n\}$ for the embedding of the period lattice bundle $\tilde{P}_{(\tilde{B}, \tilde{\mathcal{A}})} \rightarrow \tilde{B}$ of the pullback fibration $\tilde{M} \rightarrow \tilde{B}$. Note that the cell decomposition of B induces a $\pi_1(B)$ -equivariant cell decomposition of \tilde{B} ; moreover, $C_i(\tilde{B})$ is a free $\mathbb{Z}[\pi_1(B)]$ -module with basis given by choosing a representative of each of the i -cell in B . To each 2-cell e_r^2 of B , c associates a vector $\mathbf{v}_r = (v_r^1, \dots, v_r^n)$. Using the map χ , identify the vector \mathbf{v}_r with

$$v_r^1 \tilde{\theta}^1 + \dots + v_r^n \tilde{\theta}^n$$

and for each 2-cell e_r^2 in B choose a Kronecker dual $\epsilon_r^2 \in \text{hom}(C_2(B); \mathbb{Z})$. For each r , the pullback of $\epsilon_r^2 : C_2(B) \rightarrow \mathbb{Z}$ to \tilde{B} defines an element $\tilde{\epsilon}_r^2 \in \text{hom}_{\mathbb{Z}}(C_2(\tilde{B}); \mathbb{Z})$. The sum

$$\sum_{r,l} v_r^l \tilde{\epsilon}_r^2 \cup \tilde{\theta}^l$$

is then the cocycle $\Phi \circ \Psi \circ \chi(c)$, whose cohomology class (as a 3-form with real values on B) is $\mathcal{D}[c]$.

4. EXAMPLES

In this section the subgroup R of realisable Chern classes is computed for some integral affine manifolds so as to illustrate the algorithm to compute the Dazord-Delzant homomorphism \mathcal{D} of Section 3. The following theorem, quoted here without proof, is used in each example to prove that the manifolds under consideration are indeed integral affine (cf. [9]).

Theorem 4. *Let (B, \mathcal{A}) be an integral affine manifold and Γ a group acting on (B, \mathcal{A}) by integral affine diffeomorphisms such that the quotient B/Γ is a manifold. Then B/Γ inherits an integral affine structure \mathcal{A}_Γ from (B, \mathcal{A}) .*

\mathbb{T}^3 with standard integral affine structure. This example has also been considered in [11], where different methods are used. Consider the integral affine manifold $\mathbb{R}^3/\mathbb{Z}^3$, where the action of \mathbb{Z}^3 on \mathbb{R}^3 is by translations along the standard cocompact lattice. These are integral affine diffeomorphisms of \mathbb{R}^3 with respect to its standard integral affine structure and thus the quotient is an integral affine manifold, which is denoted by $\mathbb{R}^3/\mathbb{Z}^3$ for notational ease. Its linear monodromy \mathbf{l} is trivial. Let (x^1, x^2, x^3) be the standard (integral affine) coordinates on \mathbb{R}^3 inducing integral affine

coordinates on \mathbb{T}^3 , which are also denoted by (x^1, x^2, x^3) by abuse of notation. Let $F \hookrightarrow M \rightarrow \mathbb{R}^3/\mathbb{Z}^3$ be an almost Lagrangian fibration with trivial monodromy (the fibre is denoted by F to avoid confusion with the base). The period lattice bundle $P_{\mathbb{R}^3/\mathbb{Z}^3} \rightarrow \mathbb{R}^3/\mathbb{Z}^3$ associated to this fibration is trivial and the embedding of equation (8) determines a global framing of $T^*\mathbb{R}^3/\mathbb{Z}^3 \cong \mathbb{R}^3/\mathbb{Z}^3 \times \mathbb{R}^3$. Fix the framing to be $\{dx^1, dx^2, dx^3\}$. Let $\{[\beta_1], [\beta_2], [\beta_3]\}$ denote the standard ordered basis of $H^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z})$. Under the homomorphism

$$(23) \quad \begin{aligned} H^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}) &\rightarrow H^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong H^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{R}) \\ [A] &\mapsto [A] \otimes_{\mathbb{Z}} 1 \end{aligned}$$

the above basis is mapped to the standard ordered basis $\{[dx^1 \wedge dx^2], [dx^2 \wedge dx^3], [dx^3 \wedge dx^1]\}$ of $H^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{R})$. In order to follow the conventions set in [11], fix the ordered frame of the period lattice bundle to be $\{\theta^1 = dx^3, \theta^2 = dx^1, \theta^3 = dx^2\}$. Let

$$[c] = (c_{kl}) \in H^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}) \cong H^2(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{Z}) \otimes_{\mathbb{Z}} \langle dx^3, dx^1, dx^2 \rangle$$

be the Chern class of the fibration $M \rightarrow \mathbb{R}^3/\mathbb{Z}^3$, so that

$$[c] = \sum_{l,r=1}^3 c_{lr} [\beta_l] \otimes \theta^r.$$

The result of Theorem 2 shows that

$$\mathcal{D}[c] = \sum_{l,r=1}^3 c_{lr} [\beta_l^{\mathbb{R}} \cup \theta^r]$$

where $\beta_l^{\mathbb{R}}$ is a cocycle representing the image of the cohomology class $[\beta_l]$ under the map of equation (23). Therefore,

$$\mathcal{D}[c] = (c_{11} + c_{22} + c_{33}) [dx^1 \wedge dx^2 \wedge dx^3] \in H^3(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{R})$$

where the isomorphism between singular cohomology with real coefficients and de Rham cohomology has been used tacitly. Since $[dx^1 \wedge dx^2 \wedge dx^3]$ is a generator of $H^3(\mathbb{R}^3/\mathbb{Z}^3; \mathbb{R})$,

$$\mathcal{D}[c] = 0 \quad \Leftrightarrow \quad c_{11} + c_{22} + c_{33} = 0,$$

as shown in [11].

The 3-dimensional Heisenberg manifold. The 3-dimensional Heisenberg manifold is an interesting example of an integral affine manifold as its fundamental group is not abelian, but nilpotent. The relation between affine manifolds with nilpotent fundamental groups and affine geometry has already been investigated in [8], although much less is known in the context of solvable fundamental groups.

Let

$$N_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x^1 & x^3 \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix} : x^1, x^2, x^3 \in \mathbb{R} \right\}$$

denote the set of unipotent upper-triangular 3×3 matrices with real coefficients and let

$$\Gamma = N_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a^1 & a^3 \\ 0 & 1 & a^2 \\ 0 & 0 & 1 \end{pmatrix} : a^1, a^2, a^3 \in \mathbb{Z} \right\}$$

denote the lattice of unipotent upper-triangular 3×3 matrices with integral coefficients. The 3-dimensional Heisenberg manifold is given by

$$H = \Gamma \backslash N_3(\mathbb{R})$$

where Γ acts on $N_3(\mathbb{R})$ by left multiplication. $N_3(\mathbb{R}) \cong \mathbb{R}^3$ is an integral affine manifold; let $x^1, x^2, x^3 : N_3(\mathbb{R}) \rightarrow \mathbb{R}$ be globally defined integral affine coordinates on $N_3(\mathbb{R})$. The lattice $\Gamma \subset N_3(\mathbb{R})$ is generated by the matrices

$$(24) \quad \alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote a point in $N_3(\mathbb{R})$ by its integral affine coordinates (x^1, x^2, x^3) and let $L : \Gamma \rightarrow \text{Aut}(N_3(\mathbb{R}))$ denote the representation of Γ induced by left multiplication. Note that

$$(25) \quad \begin{aligned} \alpha \cdot \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \beta \cdot \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \gamma \cdot \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Equation (25) shows that the image $L(\Gamma)$ lies in the group $\text{Aff}_{\mathbb{Z}}(N_3(\mathbb{R}))$ of integral affine isomorphisms of $N_3(\mathbb{R})$. In particular, Theorem 4 implies that Heisenberg manifold H inherits an integral affine structure from its universal cover $N_3(\mathbb{R})$. Denote this integral affine manifold also by H and, by abuse of notation, let (x^1, x^2, x^3) be its integral affine coordinates. The linear monodromy $\mathfrak{l} : \Gamma = \pi_1(H) \rightarrow \text{GL}(3, \mathbb{Z})$ is defined on the generators of equation (24) by

$$(26) \quad \mathfrak{l}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathfrak{l}(\beta) = I = \mathfrak{l}(\gamma)$$

Let $M \rightarrow H$ be an almost Lagrangian fibration with monodromy \mathfrak{l}^{-T} , and let $q : N_3(\mathbb{R}) \rightarrow H$ be the universal covering map. The pullback fibration $q^*M \rightarrow N_3(\mathbb{R})$ has trivial monodromy, since $N_3(\mathbb{R})$ is simply connected. In particular, the period lattice bundle $\tilde{P}_{N_3(\mathbb{R})} \rightarrow N_3(\mathbb{R})$ into $T^*N_3(\mathbb{R})$ determines a choice of global frame of $T^*N_3(\mathbb{R})$ by closed forms. The homomorphism of equation (8) gives the ordered frame $\{\tilde{\theta}^1 = dx^3, \tilde{\theta}^2 = dx^1, \tilde{\theta}^3 = dx^2\}$ of $T^*N_3(\mathbb{R})$. This frame is *equivariant* with respect to the standard symplectic lift of the action of $\pi_1(H) = \Gamma$ on $N_3(\mathbb{R})$ by deck transformations to the cotangent bundle $T^*N_3(\mathbb{R})$.

The fundamental group Γ of H admits the following presentation in terms of generators α, β, γ of equation (24)

$$\Gamma = \langle \alpha, \beta, \gamma : \alpha\beta = \gamma\beta\alpha, \alpha\gamma = \gamma\alpha, \beta\gamma = \gamma\beta \rangle$$

H admits a CW-decomposition with one 0-cell e^0 , three 1-cells e_1^1, e_2^1, e_3^1 , three 2-cells e_1^2, e_2^2, e_3^2 and one 3-cell e^3 . This decomposition induces a $\mathbb{Z}[\Gamma]$ -equivariant CW decomposition on $\tilde{H} = N_3(\mathbb{R})$ as in the previous example. Fix a basis for the graded $\mathbb{Z}[\Gamma]$ -module $C_*(\tilde{H}) = \bigcup_{l \geq 1} C_l(\tilde{H})$, denoted by $e^0, e_1^1, e_2^1, e_3^1, e_1^2, e_2^2, e_3^2, e^3$ by abuse of notation. The boundary maps in the $\mathbb{Z}[\Gamma]$ -equivariant chain

complex are

$$\begin{aligned}
(27) \quad & \partial_1(e_1^1) = (\alpha - 1)e^0 \\
& \partial_1(e_2^1) = (\beta - 1)e^0 \\
& \partial_1(e_3^1) = (\gamma - 1)e^0 \\
& \partial_2(e_1^2) = (1 - \gamma\beta)e_1^1 + (\alpha - \gamma)e_2^1 - e_3^1 \\
& \partial_2(e_2^2) = (1 - \gamma)e_2^1 - (1 - \beta)e_3^1 \\
& \partial_2(e_3^2) = (1 - \gamma)e_1^1 + (\alpha - 1)e_3^1 \\
& \partial_3(e^3) = (\gamma - 1)e_1^2 + (\alpha - \gamma)e_2^2 + (1 - \gamma\beta)e_3^2,
\end{aligned}$$

and can be obtained from the above presentation of the group by noticing that the augmented chain complex $C_*(\tilde{H}) \rightarrow \mathbb{Z}$ is a free $\mathbb{Z}[\Gamma]$ -resolution of the trivial module \mathbb{Z} . The twisted cohomology group $H^2(H; \mathbb{Z}_{\Gamma^{-T}}^3)$ is calculated as follows. Let $\phi = (\phi_1, \phi_2, \phi_3) : C_2(\tilde{B}) \rightarrow \mathbb{Z}^3$ be a $\mathbb{Z}[\Gamma]$ -equivariant map. Then

$$\begin{aligned}
(\delta_3\phi)(e^3) &= \phi(\partial_3e^3) \\
&= (\Gamma^{-T}(\gamma) - I)\phi(e_1^2) + (\Gamma^{-T}(\alpha) - \Gamma^{-T}(\gamma))\phi(e_2^2) + (I - \Gamma^{-T}(\gamma)\Gamma^{-T}(\beta))\phi(e_3^2) \\
&= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \phi(e_2^2)
\end{aligned}$$

Thus ϕ defines a closed 2-cocycle if and only if

$$\phi_3(e_2^2) = 0,$$

and

$$(28) \quad \ker \delta \cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z} \oplus 0) \oplus (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}),$$

where each summand in brackets corresponds to the contribution of a 2-cell. Suppose now that $\phi = \delta\psi \in \text{hom}_{\mathbb{Z}[\Gamma]}(C_2(\tilde{H}); \mathbb{Z}^3)$, then

$$\begin{aligned}
(\delta\psi)(e_1^2) &= (I - \Gamma^{-T}(\gamma)\Gamma^{-T}(\beta))\phi(e_1^1) + (\Gamma^{-T}(\alpha) - \Gamma^{-T}(\beta))\phi(e_2^1) - \phi(e_3^1) \\
(\delta\psi)(e_2^2) &= (I - \Gamma^{-T}(\gamma))\phi(e_2^1) - (I - \Gamma^{-T}(\beta))\phi(e_2^1) \\
(\delta\psi)(e_3^2) &= (I - \Gamma^{-T}(\gamma))\phi(e_1^1) + (\Gamma^{-T}(\alpha) - I)\phi(e_3^1)
\end{aligned}$$

which implies that

$$(29) \quad \text{im } \delta \cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \oplus (0 \oplus 0 \oplus 0) \oplus (0 \oplus 0 \oplus 0).$$

Equations (28) and (29) imply that

$$(30) \quad H^2(H; \mathbb{Z}_{\Gamma^{-T}}^3) = \ker \delta / \text{im } \delta \cong (0 \oplus 0 \oplus 0) \oplus (\mathbb{Z} \oplus \mathbb{Z} \oplus 0) \oplus (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}).$$

Thus it is possible to calculate the image of the map \mathcal{D} using Theorem 3. Let $\epsilon_1^2, \epsilon_2^2, \epsilon_3^2 \in \text{hom}(C_2(H); \mathbb{Z})$ be Kronecker duals to e_1^2, e_2^2, e_3^2 . As explained at the end of Section 3, these induce elements $\tilde{\epsilon}_l^2 \in \text{hom}_{\mathbb{Z}}(C_2(\tilde{H}); \mathbb{Z})$ for $l = 1, 2, 3$.

Let $[c] \in H^2(H; \mathbb{Z}_{\Gamma^{-T}}^3)$ be represented by a cocycle

$$c = \sum_{l,r=1}^3 c_{lr} \tilde{\epsilon}_l^2 \otimes \tilde{\theta}^r.$$

Thus a 3-cocycle on \tilde{H} representing $\mathcal{D}[c]$ is given by

$$(31) \quad \sum_{l,r=2}^3 c_{lr} \tilde{\epsilon}_l^2 \cup \tilde{\theta}^r.$$

The calculations to determine $H^2(H; \mathbb{Z}_{[-T]}^3)$ show that it is possible to assume that $c_{1r}, c_{23} = 0$ for $r = 1, 2, 3$. Explicit representatives of $\tilde{\epsilon}_2^2, \tilde{\epsilon}_3^2$ on a fundamental domain for H in \tilde{H} in terms of differential forms are given by $dx^2 \wedge dx^3, dx^3 \wedge dx^1$ respectively. Therefore the cocycle of equation (31) can be represented in terms of differential forms by

$$(c_{22} + c_{33}) dx^1 \wedge dx^2 \wedge dx^3.$$

The form $dx^1 \wedge dx^2 \wedge dx^3$ is under the action of Γ on \tilde{H} and so it descends to H ; its cohomology class $[dx^1 \wedge dx^2 \wedge dx^3]_H$ generates $H^3(H; \mathbb{R})$ and therefore

$$\mathcal{D}[c] = (c_{22} + c_{33}) [dx^1 \wedge dx^2 \wedge dx^3]_H.$$

Mapping torus of $-\text{Id}$. $\mathbb{T}^2 \rightarrow \mathbb{T}^2$. Let B denote the mapping torus of the involution $-\text{Id} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$. This manifold admits an integral affine structure, since it can be realised as follows. Consider the $\mathbb{Z}/2$ -action on $\mathbb{R}^3/\mathbb{Z}^3$ defined by

$$(32) \quad \zeta \cdot (x^1, x^2, x^3) = (x^1 + 1/2, -x^2, -x^3)$$

where (x^1, x^2, x^3) are integral affine coordinates on \mathbb{T}^3 as in the previous example, and $\zeta \in \mathbb{Z}/2$ is a generator. The action is free and by integral affine transformations and thus the quotient $B = (\mathbb{R}^3/\mathbb{Z}^3)/(\mathbb{Z}/2)$ is an integral affine manifold. Let (B, \mathcal{A}) denote this integral affine manifold and let \mathfrak{l} be its linear holonomy.

Let $F \hookrightarrow M \rightarrow (B, \mathcal{A})$ be an almost Lagrangian fibration with monodromy \mathfrak{l}^{-T} and Chern class $[c] \in H^2(B; \mathbb{Z}_{[-T]}^3)$. Denote by $q : \mathbb{R}^3/\mathbb{Z}^3 \rightarrow B$ the integral affine double covering map given by the $\mathbb{Z}/2$ -action of equation (32). The pullback almost Lagrangian fibration $q^*M \rightarrow \mathbb{R}^3/\mathbb{Z}^3$ has trivial monodromy by construction. The period lattice bundle $\hat{P}_{\mathbb{R}^3/\mathbb{Z}^3} \rightarrow \mathbb{R}^3/\mathbb{Z}^3$ associated to the pullback fibration $q^*M \rightarrow \mathbb{R}^3/\mathbb{Z}^3$ is trivial and its embedding into $T^*\mathbb{R}^3/\mathbb{Z}^3$ induces a choice of global ordered frame $\{\tilde{\theta}^1 = dx^3, \tilde{\theta}^2 = dx^1, \tilde{\theta}^3 = dx^2\}$.

The fundamental group of B can be presented using generators and relations as

$$\pi = \pi_1(B) = \langle \alpha, \beta, \gamma : \beta\gamma = \gamma\beta, \alpha = \beta\alpha\beta, \alpha = \gamma\alpha\gamma \rangle.$$

The subgroup generated by α^2, β, γ is isomorphic to \mathbb{Z}^3 and corresponds to the fundamental group of the double cover $\mathbb{R}^3/\mathbb{Z}^3 \rightarrow B$. The linear monodromy of the integral affine manifold B is given by the map $\mathfrak{l} : \pi_1(B) \rightarrow \text{GL}(3, \mathbb{Z})$ defined on the generators by

$$(33) \quad \mathfrak{l}(\alpha) = \text{diag}(-1, 1, -1), \quad \mathfrak{l}(\beta) = I, \quad \mathfrak{l}(\gamma) = I,$$

where the coordinate system used is (x^3, x^1, x^2) , in line with the above choice of ordered frame of $T^*\mathbb{R}^3/\mathbb{Z}^3$. The manifold B admits a CW decomposition with one 0-cell e^0 , three 1-cells e_1^1, e_2^1, e_3^1 , three 2-cells e_1^2, e_2^2, e_3^2 and one 3-cell e^3 . This decomposition induces a π -equivariant cell decomposition on the universal cover $\tilde{B} = \mathbb{R}^3$

$$\mathbb{R}^3 = \bigcup_{g \in \pi_1(B)} (e_g^0 \cup e_{1,g}^1 \cup e_{2,g}^1 \cup e_{3,g}^1 \cup e_{1,g}^2 \cup e_{2,g}^2 \cup e_{3,g}^2 \cup e_g^3)$$

Fix the following $\mathbb{Z}[\pi]$ -basis for the module $C_i(\tilde{B})$

$$(34) \quad \begin{cases} e_{g_0}^0 & \text{if } i = 0 \\ e_{1,g_0}^1, e_{2,g_0}^1, e_{3,g_0}^1 & \text{if } i = 1 \\ e_{1,g_0}^2, e_{2,g_0}^2, e_{3,g_0}^2 & \text{if } i = 2 \\ e_{g_0}^3 & \text{if } i = 3 \end{cases}$$

where $g_0 \in \pi$ is some fixed element. For notational ease, suppress the dependence on g_0 . The boundary maps

$$\partial_i : C_i(\tilde{B}) \rightarrow C_{i-1}(\tilde{B})$$

of the $\mathbb{Z}[\pi]$ -equivariant chain complex of B are given on the basis of equation (34) by

$$(35) \quad \begin{aligned} \partial_1(e_1^1) &= (\alpha\beta\gamma - 1)e^0 \\ \partial_1(e_2^1) &= (\beta - 1)e^0 \\ \partial_1(e_3^1) &= (\gamma - 1)e^0 \\ \partial_2(e_1^2) &= (1 - \beta)e_1^1 - (1 + \alpha\gamma)e_2^1 \\ \partial_2(e_2^2) &= (1 - \gamma)e_2^1 - (1 - \beta)e_3^1 \\ \partial_2(e_3^2) &= (1 - \gamma)e_1^1 - (1 + \alpha\beta)e_3^1 \\ \partial_3(e^3) &= (1 - \gamma)e_1^2 - (1 - \alpha)e_2^2 + (1 - \beta)e_3^2 \end{aligned}$$

using standard methods in algebraic topology. The corresponding $\mathbb{Z}[\pi]$ -equivariant cochain complex with values in \mathbb{Z}^3 arises when applying the equivariant $\text{hom}_{\mathbb{Z}[\pi]}(\cdot; \mathbb{Z}^3)$ functor to the chain complex defined above. In particular, for the calculation of $H^2(B; \mathbb{Z}_{\Gamma^{-T}}^3)$, the relevant maps are

$$\text{hom}_{\mathbb{Z}[\pi]}(C_1(\tilde{B}); \mathbb{Z}^3) \xrightarrow{\delta_2} \text{hom}_{\mathbb{Z}[\pi]}(C_2(\tilde{B}); \mathbb{Z}^3) \xrightarrow{\delta_3} \text{hom}_{\mathbb{Z}[\pi]}(C_3(\tilde{B}); \mathbb{Z}^3)$$

The twisted cohomology group $H^2(B; \mathbb{Z}^3)$ is given by

$$H^2(B; \mathbb{Z}_{\Gamma^{-T}}^3) = \ker \delta_3 / \text{im } \delta_2.$$

Let $\phi : C_2(\tilde{B}) \rightarrow \mathbb{Z}^3$ be a twisted cocycle, then

$$\begin{aligned} 0 &= (\delta_3\phi)(e^3) = \phi(\partial_3 e^3) \\ &= (I - \Gamma^{-T}(\gamma))\phi(e_1^2) - (I - \Gamma^{-T}(\alpha))\phi(e_2^2) + (I - \Gamma^{-T}(\beta))\phi(e_3^2) \\ &= \text{diag}(2, 0, 2)\phi(e_2^2) \end{aligned}$$

where the last equality follows from the definition of Γ^{-T} in equation (33). Thus ϕ is a twisted cocycle if and only if

$$(36) \quad \phi_1(e_2^2) = 0 = \phi_3(e_2^2)$$

where $\phi = (\phi_1, \phi_2, \phi_3)$. Suppose now that ϕ is a twisted coboundary, *i.e.* that it lies in the image of δ_2 , so that $\phi = \delta_2\psi$. Using the boundary maps of equation (35), it is possible to obtain the following conditions on ϕ

$$(37) \quad \phi_2(e_1^2) = 2k_1, \quad \phi_2(e_3^2) = 2k_2$$

for $k_1, k_2 \in \mathbb{Z}$. Using equations (36) and (37), it follows that

$$H^2(B; \mathbb{Z}_{\Gamma^{-T}}^3) \cong (\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}) \oplus (0 \oplus \mathbb{Z} \oplus 0) \oplus (\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}).$$

Choose Kronecker duals $e_l^2 \in \text{hom}(C_2(B); \mathbb{Z})$ for each 2-cells e_l^2 in B . These induce elements $\tilde{e}_l^2 \in \text{hom}_{\mathbb{Z}}(C_2(\mathbb{R}^3/\mathbb{Z}^3); \mathbb{Z})$. Let c be an equivariant cocycle representing $[c] \in H^2(B; \mathbb{Z}_{l-T}^3)$ on $\mathbb{R}^3/\mathbb{Z}^3$. It takes the form

$$c = \sum_{l,r=1}^3 c_{lr} \tilde{e}_l^2 \cup \tilde{\theta}^r$$

Explicit representatives of $\tilde{e}_1^2, \tilde{e}_2^2, \tilde{e}_3^2$ on a fundamental domain for B in $\mathbb{R}^3/\mathbb{Z}^3$ in terms of differential forms are given by $dx^1 \wedge dx^2, dx^2 \wedge dx^3, dx^3 \wedge dx^1$ respectively. Therefore the above cocycle can be represented in terms of differential forms on $\mathbb{R}^3/\mathbb{Z}^3$ as

$$(c_{11} + c_{22} + c_{33}) dx^1 \wedge dx^2 \wedge dx^3.$$

Since $dx^1 \wedge dx^2 \wedge dx^3$ is invariant under the action of π by deck transformations on $\mathbb{R}^3/\mathbb{Z}^3$, it descends to a closed 3-form on B whose cohomology class $[dx^1 \wedge dx^2 \wedge dx^3]_B$ generates $H^3(B; \mathbb{R})$. Thus

$$\mathcal{D}[c] = (c_{11} + c_{22} + c_{33}) [dx^1 \wedge dx^2 \wedge dx^3]_B.$$

5. CONCLUSION

The above description of the map \mathcal{D} further emphasises the relation between (almost) Lagrangian fibrations and integral affine manifolds. While being useful to carry out the classification of Lagrangian fibrations over higher dimensional manifolds, the map \mathcal{D} should also reveal information about integral affine manifolds. For instance, a natural question to ask is

Question. Is it possible to choose the forms $\tilde{\theta}^1, \dots, \tilde{\theta}^n$ so that the image of the map \mathcal{D} lies in a cohomology group $H^3(B; S)$, where S is some finite ring extension of the integers?

The examples of Section 4 show that this might indeed be the case. In order to answer this and many other related questions, it is necessary to investigate what it means for a manifold to be integrally affine, which is central to the study of the topology and symplectic geometry of completely integrable Hamiltonian systems.

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